THE EFFECT OF POLAR STEREOGRAPHIC PROJECTION ON THE CALCULATION OF THE CURVATURE OF HORIZONTAL CURVES*

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ABSTRACT

It is shown that, in any secant polar stereographic projection, a small circle on a sphere projects into a circle. This property provides a simple relationship between K_H, the horizontal component of curvature of a horizonal curve and K'_H, the curvature of its projection on a secant polar stereographic map. K_H can be computed by subtracting from the map factor times K'_H the earth's curvature multiplied by a correction factor that depends only on the latitude of the place and inclination of the curve to the latitude circle. This factor vanishes if the curve is along a meridian but takes an extreme value if it is along a latitude. For a given orientation of the curve, the value of this factor increases gradually as the location of the curve moves from the Pole to the Equator and more rapidly after it crosses the Equator. It is less than 1 in the Northern Hemisphere but can exceed unity in the Southern Hemisphere.

1. INTRODUCTION

The polar stereographic projection is used for several purposes, especially for weather charts in the middle and higher latitudes. Sometimes it is of interest to compute the curvature of certain contours as well as other types of curves like isobars from the curvature of their projections on this chart. The following derivation establishes a relationship between the curvatures of a horizontal curve and its projection on a polar stereographic chart.

2. DERIVATION

In figure 1, point P (colatitude ψ and longitude λ) lies on a circle with its center below point C (colatitude ψ_0). The longitude of C can be taken as 0° without loss of generality. The angular radius, b, of the circle is given by

$$\cos b = \cos \psi_0 \cos \psi + \sin \psi_0 \sin \psi \cos \lambda. \tag{1}$$

In circumpolar stereographic projection with scale true at colatitude ψ_1, the radius of projection of any latitude circle is

$$r = c \tan \psi/2 \tag{2}$$

where $c = a(1+\cos \psi_1)$, a being the radius of the earth. Eliminating ψ between (1) and (2)

$$r^2 - 2rd \cos \lambda - c^2 \frac{\cos \psi_0 - \cos b}{\cos \psi_0 + \cos b} = 0 \tag{3}$$

which is a quadratic in r representing, in polar coordinates, a circle of radius,

$$r'_H = c \frac{\sin b}{(\cos \psi_0 + \cos b)} \tag{4}$$

with center at distance from the Pole,

$$d = c \frac{\sin \psi_0}{(\cos \psi_0 + \cos b)}. \tag{5}$$

Equation (3) proves that a small circle on a sphere projects into a circle on the projection in question. The curvature of the latter circle is

$$K'_H = \frac{\cos \psi_0 + \cos b}{c \sin b}.$$

Taking the spherical triangle NCP (fig. 1), it can be shown that

$$K_H = m\left(K'_H - \frac{\sin \psi \cos \beta}{a(1+\cos \psi_1)}\right) \tag{6}$$

where K_H is the horizontal curvature of the original circle, m is the scale factor, and β is the angle between the curve and the latitude circle.

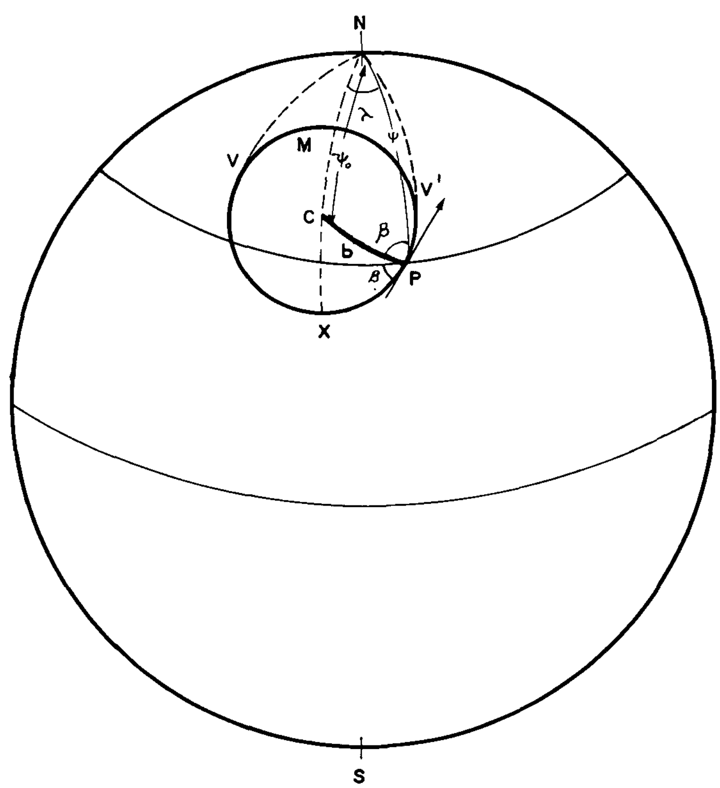

FIGURE 1.—The coordinates of a point on a small circle on a sphere.

*These results were presented at a meeting of the Poona Branch of the Indian Meteorological Society in 1960 in the course of a talk on "Conformal Map Projections Used in Meteorology."

The values of K_H and m are

$$K_H=\frac{1}{a}\cot b \text{ and } m=\frac{(1+\cos\psi_1)}{(1+\cos\psi)}.$$

Equation (6) provides a relationship between K_H and K'_H in terms of m, β, ψ, ψ_1. In the case of a tangent polar stereographic projection ($\psi_1=0$), this reduces to

$$K_H=m\left(K'_H-\frac{\sin\psi\cdot\cos\beta}{2a}\right) \tag{7}$$

as given by Haltiner and Martin [1].

Equation (6) provides a value for the error E, the difference between the curvature of the curve and curvature of its projection adjusted for scale.

$$E=K_H-mK'_H=-\frac{1}{a}\tan\frac{\psi}{2}\cos\beta. \tag{8}$$

E is, however, independent of ψ_1, the location of the standard parallel. It vanishes at points V and V′ (see fig. 1) where the curve touches the meridian. Its magnitude has maximum and minimum values at points X and M respectively, i.e., when the curve touches a latitude circle. The magnitude of the extreme value of the error is $\tan\psi/2$ times the curvature of the earth. Thus, in the Northern Hemisphere (for projection from the South Pole) where ψ is less than 90°, the error is always less than the curvature of the earth and it increases as the latitude decreases. As the location moves into the Southern Hemisphere, it can exceed unity.

3. DISPLACEMENT OF THE CENTER OF CURVATURE

Whereas a circle projects into a circle, its center does not project into the center of its projection. The center of the projection of the circle lies farther away from the Pole than the projection of the center of the circle.

Figure 2 represents a cross section of the sphere shown in figure 1 along the meridional plane passing through C. The diameter MX of the small circle projects into M′X′ and the center C into C′. P′ is the midpoint of M′X′. It is evident that P′ and C′ are not one and the same, since MC and CX will not project into equal lengths. If the angular shift of the center of the circle (viz CP) is α degrees from the Pole, from (2) and (5)

$$\tan\frac{\psi_0+\alpha}{2}=\frac{\sin\psi_0}{(\cos\psi_0+\cos b)}.$$

Expanding the above equation in terms of tangents of half angles of α, b, and ψ_0 and solving for $\tan\alpha/2$

$$\tan\frac{\alpha}{2}=\tan\frac{\psi_0}{2}\tan^2\frac{b}{2}. \tag{9}$$

If α and b are expressed in degrees we have, for small values of b, the approximate relationship,

$$\alpha\approx\frac{b^2}{100}\tan\frac{\psi_0}{2}. \tag{10}$$

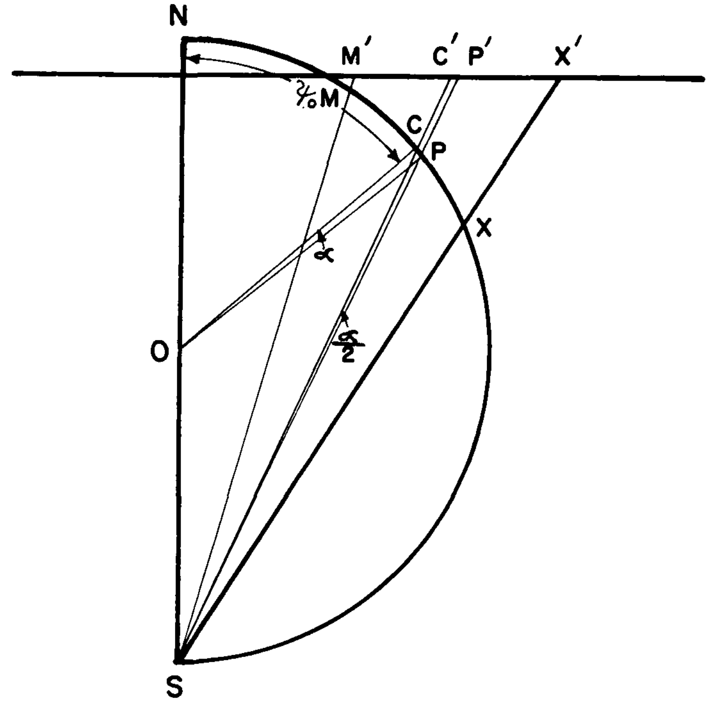

FIGURE 2.—Cross section of a sphere, a small circle and its projection.

If the center of the circle lies on the Equator the apparent shift of the center of the projection of a circle is approximately 15′ and 1° for circles of radius 5° and 10° respectively. The corresponding values for circles with centers at 30°N. lat. are 9′ and 35′ respectively.

Since the center of the projected circle is not the same as the projection of the center of the small circle, concentric small circles other than latitude circles will not project into concentric circles. The centers of the projection of these concentric circles will be different and will lie on the same meridian; the center of a circle with larger radius will be displaced farther away from Pole.

The complications introduced by the distortions due to the projection can be avoided if the diameter of the circle of curvature is taken as the difference in the latitudes of the points M and X where the meridian through the center intersects the circle of curvature and K_H is calculated from its definition (see equation (6)).

ACKNOWLEDGMENT

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REFERENCE

1. G. J. Haltiner and F. L. Martin, *Dynamical and Physical Meteorology*, McGraw-Hill Book Company, Inc., New York, 1957, 470 pp. (see p. 175).

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PICTURE OF THE MONTH

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Mesoscale cloud patterns are strongly influenced by the terrain features of an area. A frequently observed example is the formation of wave clouds in the area of gravity waves to the lee of mountains. These clouds occur when: the wind direction is perpendicular to the mountains through a deep layer, the mountain top wind is a minimum of 20 kt., and the atmosphere is stable for vertical displacements of air. Satellite photographs show that the unique parallel arrangement of small wave clouds is common to all major mountain chains throughout the world. In the United States, lee waves are frequently observed, as in this case, along the northwestern ranges.

On June 21, 1968, the 1200 GMT analysis showed a weak surface High centered in Wyoming with a low pressure area off the Washington coast. The 500- and 300-mb. analysis showed strong zonal flow across the northwestern United States. At this time, the 200-mb. jet stream was analyzed to cross the coast near Seattle and follow a path, due east, along 48°N. through Idaho and Montana and then northeastward into Canada.

Upper air soundings for 1200 GMT at Lander, Wyo. (LND), and Great Falls, Mont. (GTF), accompany the 1435 GMT ESSA-2 photograph in figure 1. At this time, low clouds are present near LND and middle and high clouds at GTF. Little directional wind shear is indicated at both these stations.

The ESSA-5 picture (fig. 2), taken at 2309 GMT, shows that late morning and early afternoon convection in this area has resulted in a large area of wave clouds through Idaho, Montana, and Wyoming. The 0000 GMT soundings at both stations show the lapse rate to be dry adiabatic up to the base of the inversion. At GTF the winds aloft have increased due to a shift in the jet stream; now entering the coast at 49°N., it continues eastward to 115°W. and gradually turns southeastward passing through the extreme southwest corner of North Dakota.

The brightest group of wave clouds (Q) is found near the 6,000- to 9,000-ft. Cabinet Mountains and the Bitter Root Range. The clouds become more widely spaced to the east in the vicinity of the Rocky Mountains of Montana. The upper air data at GTF indicates that the wind at the mountain top level is greater than 26 kt. from the west-southwest.

The wave cloud pattern to the south (R) is in the vicinity of the higher Rocky Mountains in Wyoming and

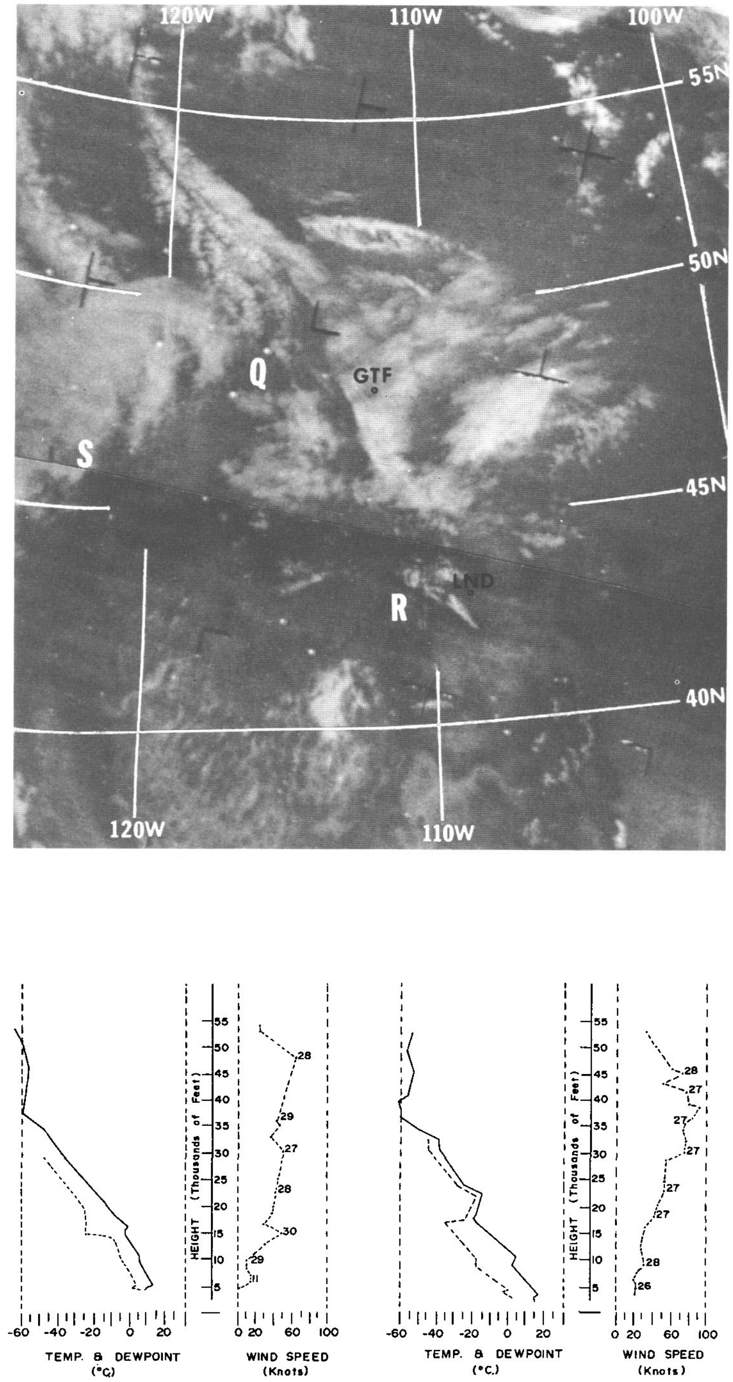
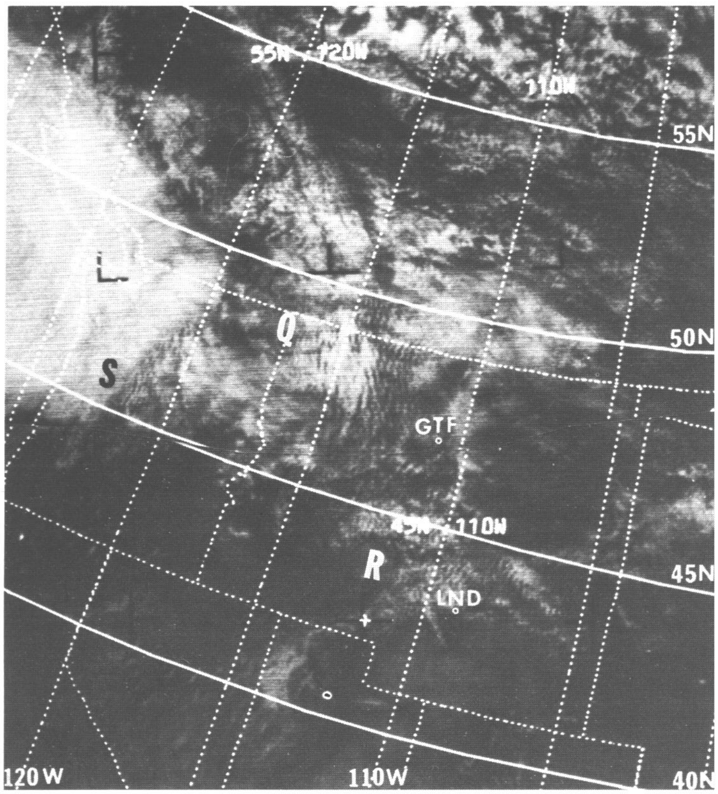


FIGURE 1.—ESSA 2, APT, Orbit 10705, 1436 GMT and 1200 GMT, upper air soundings for Lander, Wyo. (LND), and Great Falls, Mont. (GTF), June 21, 1968.



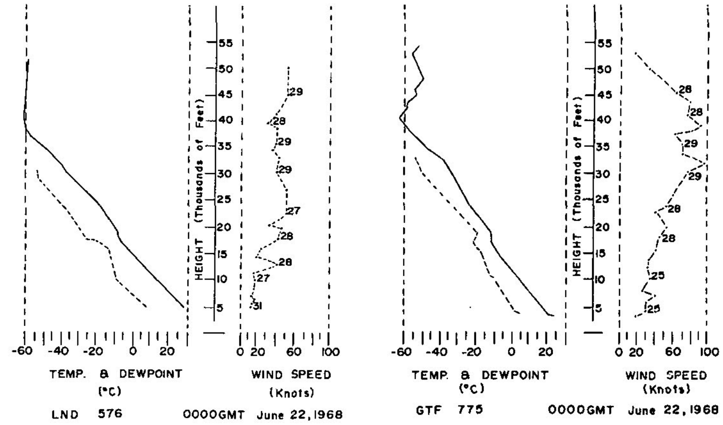


FIGURE 2.—ESSA 5, Orbit 5431, 2309 GMT, June 21, 1968, and 0000 GMT, upper air soundings for Lander, Wyo. (LND), and Great Falls, Mont. (GTF), June 22, 1968.

southern Montana. These clouds lie to the north of Yellowstone Park, and along the north-south Absaroka and Wind River Ranges, and farther east along the Big Horn Mountains. The wind at the top of the 12,000- and 14,000-ft. mountains, indicated by LND, is 40 kt. from the west.

Another area of wave clouds (S) can be seen along the eastern edge of the frontal cloudiness approaching Washington and Oregon, in the vicinity of Mt. Adams and Mt. Hood.

The presence of wave clouds in satellite photographs provides the aviation forecaster with visual information about the mesoscale wind patterns and general atmospheric structure in the vicinity of mountainous areas. Although the distribution of turbulence associated with lee waves is still under investigation, some preliminary results indicate that the turbulent layer is confined to the area within and below these clouds. Soaring and glideplane pilots were among the first to investigate wave clouds, and by flying these clouds, these pilots have established new height and distance records. Using the same technique, light aircraft pilots have found that they can conserve fuel by "riding" the wave clouds.